The MAJOR ideas of module 3

- **Area, volume, and their properties**
- **Dissection**
- **Cavalieri's Principle**
- **Applications**

Included ideas

- **Unit conversions**
- **Pythagorean Theorem**
- **Trig ratios (right triangle)**
- **Similar figures length ratios**
- **Visualization**

Table of contents:

The reference page . . . you're gonna need it!

Common Core High School Math Reference Sheet (Algebra I, Geometry, Algebra II)

CONVERSIONS

1 inch = 2.54 centimeters	1 kilometer = 0.62 mile	1 cup = 8 fluid ounces
1 meter = 39.37 inches	1 pound = 16 ounces	1 pint = 2 cups
1 mile = 5280 feet	1 pound = 0.454 kilograms	1 quart = 2 pints
1 mile = 1760 yards	1 kilogram = 2.2 pounds	1 gallon = 4 quarts
1 mile = 1.609 kilometers	1 ton = 2000 pounds	1 gallon = 3.785 liters
		1 liter = 0.264 gallon
		1 liter = 1000 cubic centimeters

FORMULAS

Triangle	$A = \frac{1}{2}bh$	Pythagorean Theorem	$a^2 + b^2 = c^2$
Parallelogram	$A = bh$	Quadratic Formula	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Circle	$A = \pi r^2$	Arithmetic Sequence	$a_n = a_1 + (n-1)d$
Circle	$C = \pi d$ or $C = 2\pi r$	Geometric Sequence	$a_n = a_1 r^{n-1}$
General Prisms	$V = Bh$	Geometric Series	$S_n = \frac{a_1 - a_1 r^n}{1 - r}$ where $r \neq 1$
Cylinder	$V = \pi r^2 h$	Radians	$1 \text{ radian} = \frac{180}{\pi} \text{ degrees}$
Sphere	$V = \frac{4}{3}\pi r^3$	Degrees	$1 \text{ degree} = \frac{\pi}{180} \text{ radians}$
Cone	$V = \frac{1}{3}\pi r^2 h$	Exponential Growth/Decay	$A = A_0 e^{k(t - t_0)} + B_0$
Pyramid	$V = \frac{1}{3}Bh$		

The standards

G-GMD.A.1 Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. *Use dissection arguments, Cavalieri's principle, and informal limit arguments.*

G-GMD.A.3 Use volume formulas for cylinders, pyramids, cones and spheres to solve problems.★

G-GMD.B.4 Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.

G-MG.A.1 Use geometric shapes, their measures, and their properties to describe objects (e.g. modeling a tree trunk or a human torso as a cylinder).★

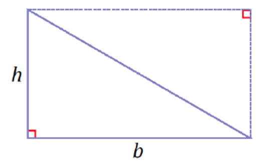
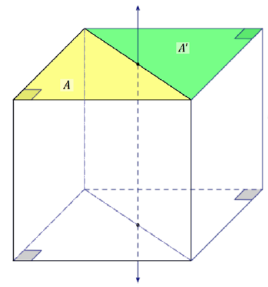
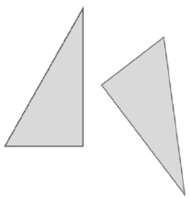
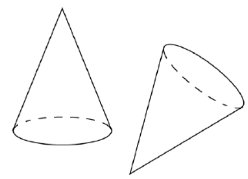
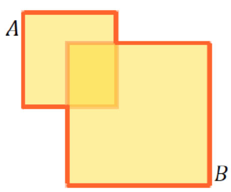
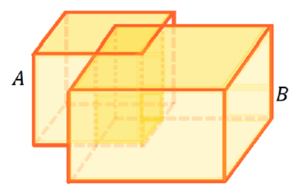
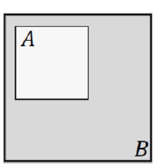
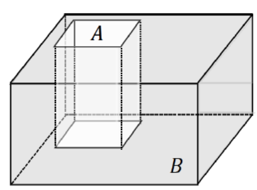
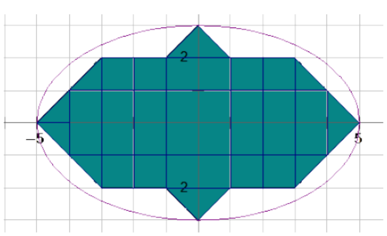
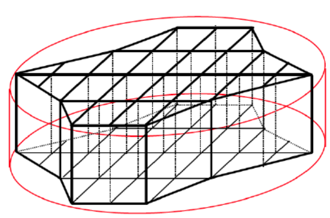
G-MG.A.2 Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot).★

G-MG.A.3 Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios).★

G-GMD.A.2 (+) Give an informal argument using Cavalieri's principle for the formulas for the volume of a sphere and other solid figures.

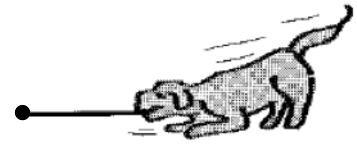
MP.6 **Attend to precision**.

MP.7 **Look for and make use of structure**.

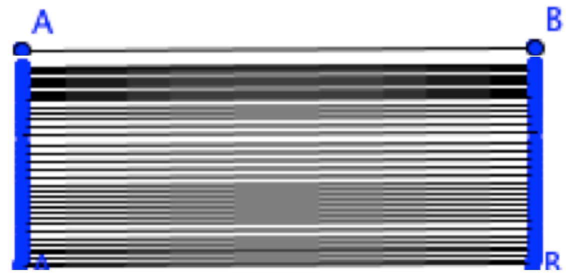
Area Properties	Volume Properties
1. The area of a set in two dimensions is a number greater than or equal to zero that measures the size of the set and not the shape.	**1.** The volume of a set in three dimensions is a number greater than or equal to zero that measures the size of the set and not the shape.
2. The area of a rectangle is given by the formula $\text{length} \times \text{width}$. The area of a triangle is given by the formula $\frac{1}{2}\,\text{base} \times \text{height}$. A polygonal region is the union of finitely many non-overlapping triangular regions and has area the sum of the areas of the triangles.  	**2.** A right rectangular or triangular prism has volume given by the formula $\text{area of base} \times \text{height}$. A right prism is the union of finitely many non-overlapping right rectangular or triangular prisms and has volume the sum of the volumes of the prisms. 
3. Congruent regions have the same area. 	**3.** Congruent solids have the same volume. 
4. The area of the union of two regions is the sum of the areas minus the area of the intersection: $\text{Area}(A \cup B) = \text{Area}(A) + \text{Area}(B) - \text{Area}(A \cap B)$  	**4.** The volume of the union of two solids is the sum of the volumes minus the volume of the intersection: $\text{Vol}(A \cup B) = \text{Vol}(A) + \text{Vol}(B) - \text{Vol}(A \cap B)$ 
5. The area of the difference of two regions where one is contained in the other is the difference of the areas: If $A \subseteq B$, then $\text{Area}(B - A) = \text{Area}(B) - \text{Area}(A)$.  	**5.** The volume of the difference of two solids where one is contained in the other is the difference of the volumes: If $A \subseteq B$, then $\text{Vol}(B - A) = \text{Vol}(B) - \text{Vol}(A)$. 
6. The area a of a region A can be estimated by using polygonal regions S and T so that S is contained in A and A is contained in T. Then $\text{Area}(S) \le a \le \text{Area}(T)$.  	**6.** The volume v of a solid W can be estimated by using right prism solids S and T so that $S \subseteq W \subseteq T$. Then $\text{Vol}(S) \le v \le \text{Vol}(T)$. 

(1) The concept of area and use of a grid to quantify it

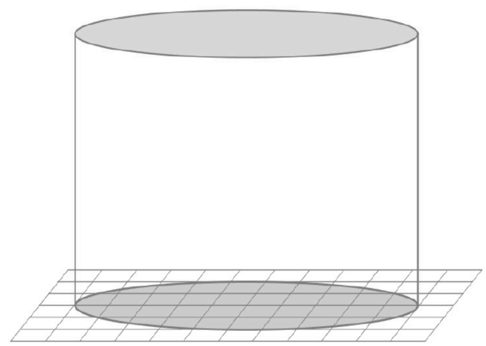
By dragging a point, we get a line . . . something to measure.



By dragging a line segment, we get an area . . . something new to measure. Area is the quantity of the "set."

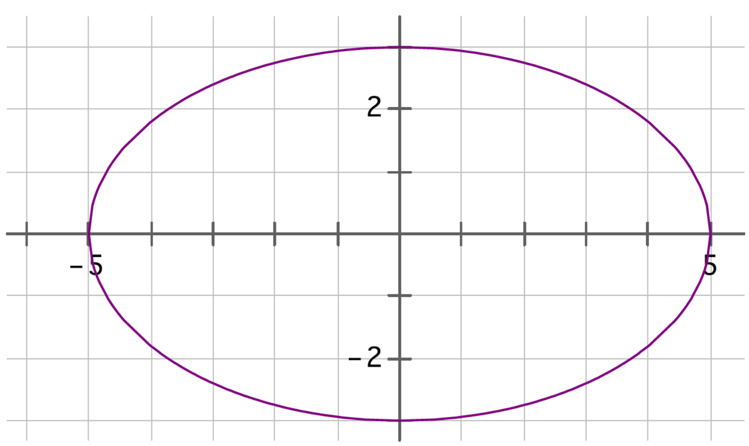


By dragging an area off the plane, we get a volume . . . something new to measure. Volume is the quantity of the "set."

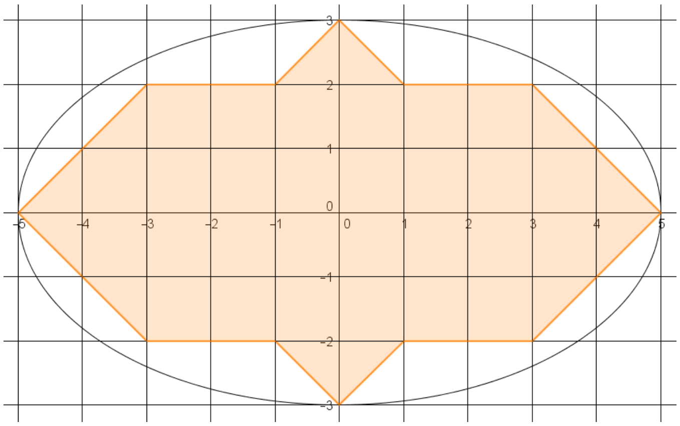


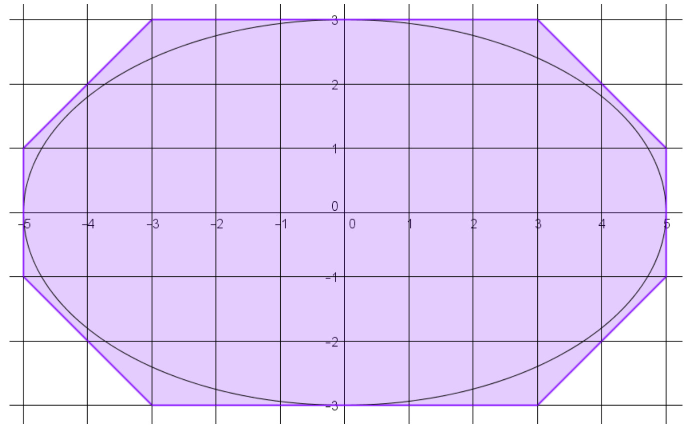
Module lesson #1 page 7 question

If it takes one can of paint to cover a unit square in the coordinate plane, how many cans of paint are needed to paint the region within the curved figure?

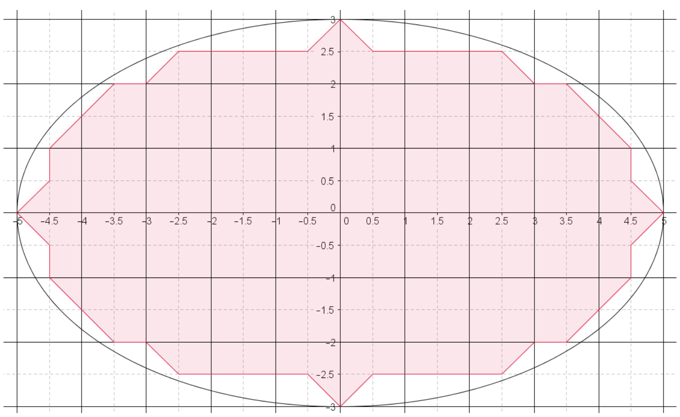


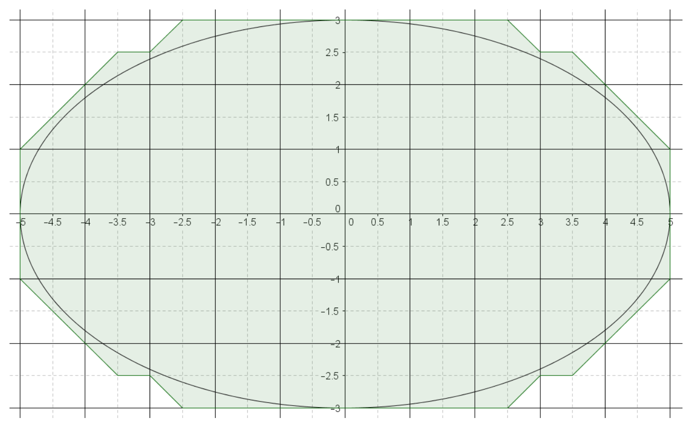
How do the pair of diagrams below help?
Make a conclusion about the area of the original shape based on these two shaded areas.





How do the pair of diagrams below help us improve our conclusion from the last pair of diagrams?



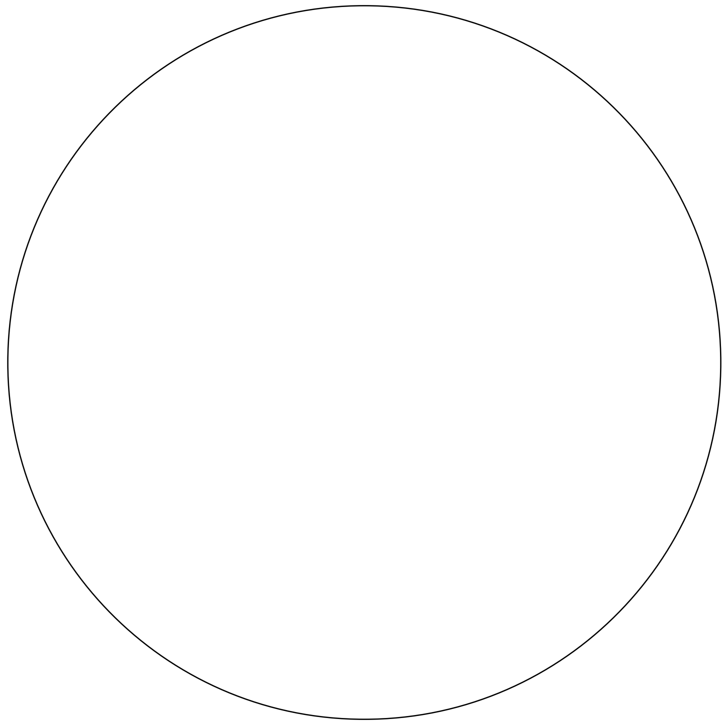


What if we continue using a smaller and smaller grid? Can we **squeeze** the area out of squares on a grid by reaching a limit? (Note: lesson 8 shows this again for a prism.)

(2) Using limits to "squeeze" area

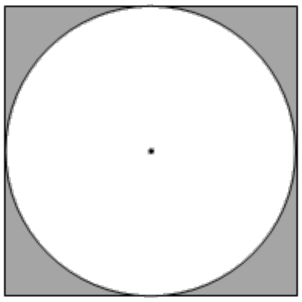
Module lesson #4 page 50 question

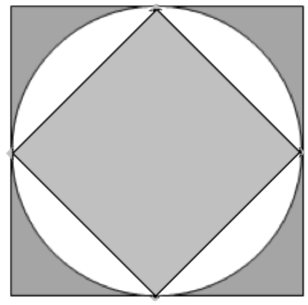
How can we use the idea of limit to get the area formula for a circle?



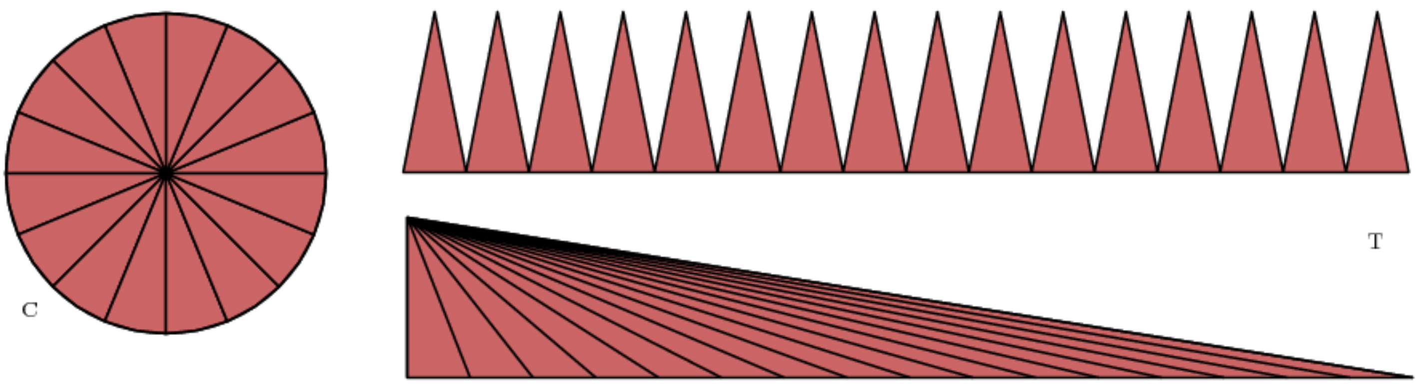
Some ideas for “squeezing circle area.

Idea #1. Use r to represent the radius of the circle. How can the area of the circle be “squeezed” between the areas of the outer square and the inner square?

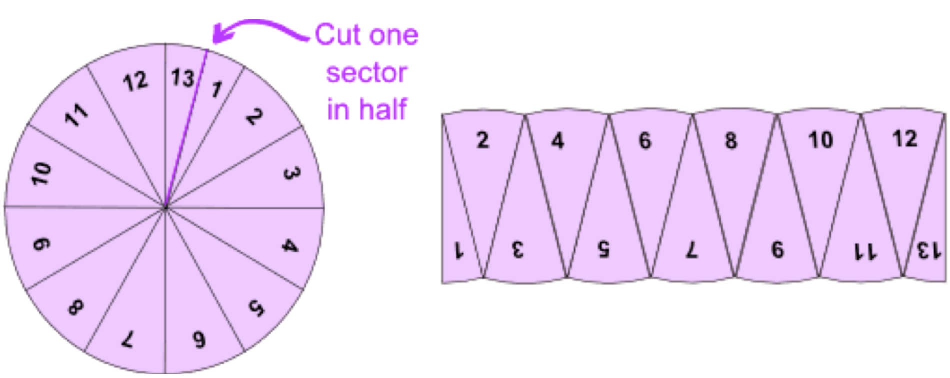




Idea #2. From: http://www.ams.org/samplings/feature-column/fc-2012-02
Using the fact that the circumference of a circle is 2π, Archimedes uses the diagram below to show that the area of a circle is πr^2. How can Archimedes’ diagrams be used to show that the area of a circle is πr^2 ?



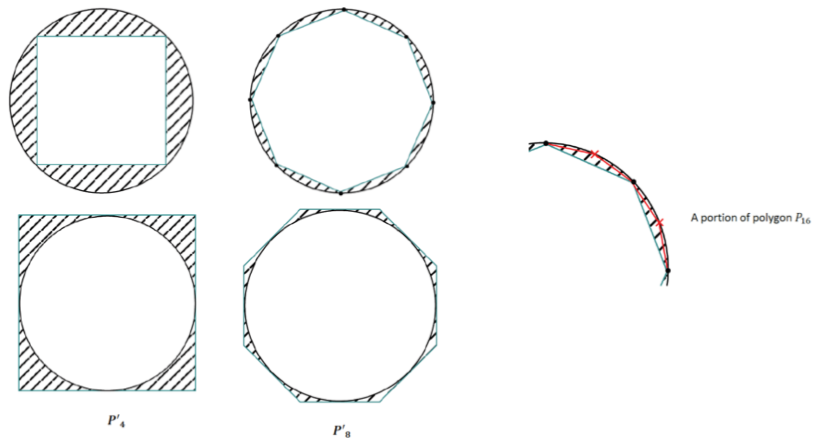
Idea #3. From http://www.mathsisfun.com/geometry/circle-area-by-sectors.html
Using the fact that the circumference is 2π, how can we use the diagram below to show the area of the circle is πr^2?



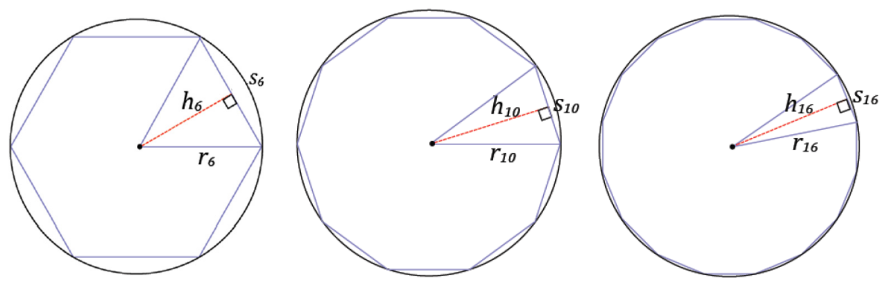
Module lesson #4 page 51 question

Idea #4. From Module 3

By doubling the sides of a polygon inside and outside of the circle, the area of the circle gets squeezed between the polygons. To do this, polygon areas and limits are introduced.

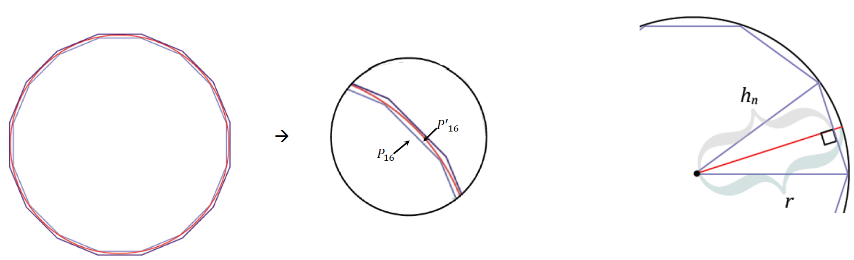


Based on the diagram at right, how can we find the area of the regular inscribed polygons?



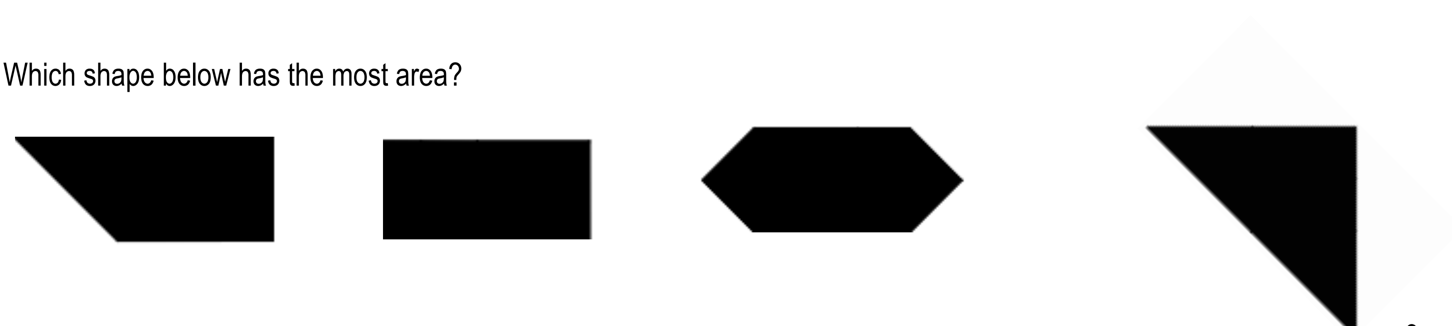
How can we simplify the formula to work for any number of sides?

By squeezing the circle with polygons, the limit for the perimeter as n (the number of sides) approaches infinity is the circumference of the circle or $2\pi r$. Therefore, the area of the circle can be written using this limit. Write and simplify the equation for the area of the circle.

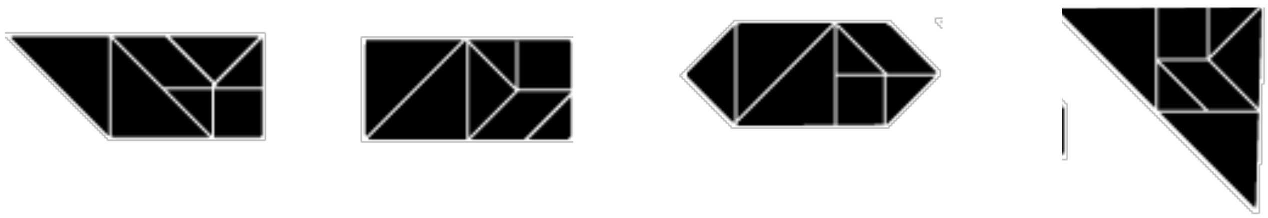


(3) Dissection (including applications)

Which shape below has the most area?

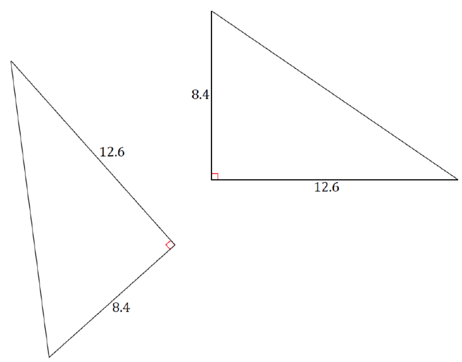


The same, because they are made from the same set of tangrams. Area is a quantity, not a shape.

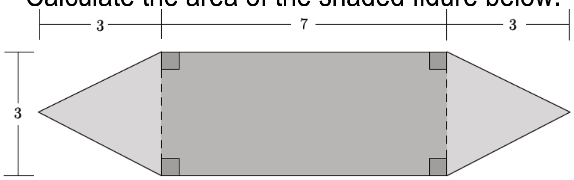


Examples of area "sets"
Module lesson #2 page 25 questions

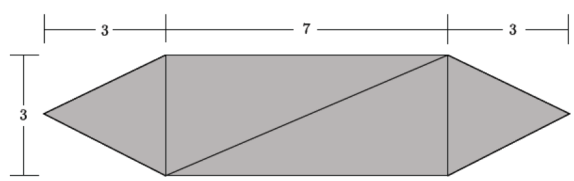
Calculate the area of the 2 congruent triangles.
Of the 4 transformations we have studied,
which ones will result in congruent triangles?



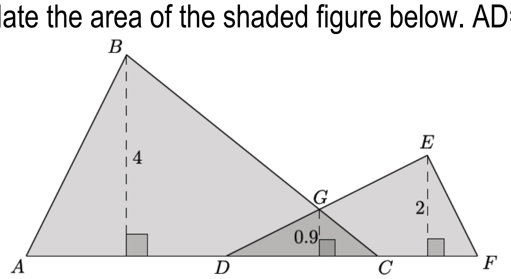
Calculate the area of the shaded figure below.



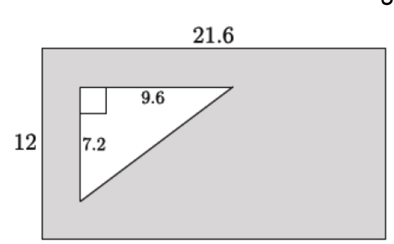
Would you get the same result by finding the area of the triangles?



Calculate the area of the shaded figure below. AD=4, DC=3, CF=2

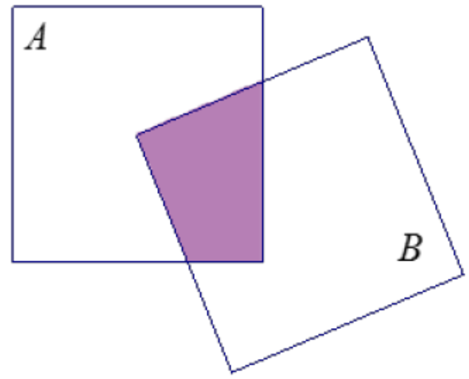


Calculate the are of the shaded figure below.



CHALLENGE!
Module lesson #2 page 34 question

Two square regions A and B each have an area of 8 square units. One vertex of B is the center of A. Can you find the shaded area and the area of the outer boundary formed by A and B together without any further information? What are the possible areas?



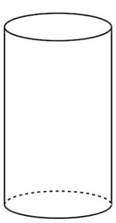
Dissections in 3D

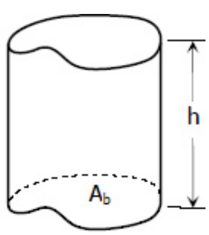
First, what do you think is meant by a "general cylinder?" (describe or draw)

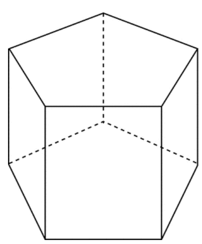
Module lesson #6 page 85 question
Here, maybe this from the module will help:

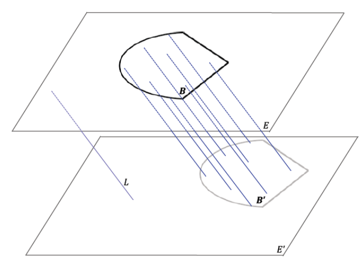
- **General Cylinder** (Let E and E' be two parallel planes, let B be a region in the plane E, and let L be a line which intersects E and E' but not B. At each point P of B, consider the segment PP' parallel to L, joining P to a point P' of the plane E'. The union of all these segments is called a *cylinder with base* B.)

In the event that the definition above did not help, the figures below are all examples of general cylinders.



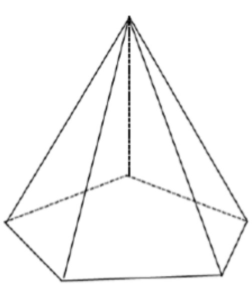


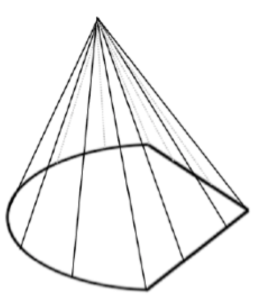


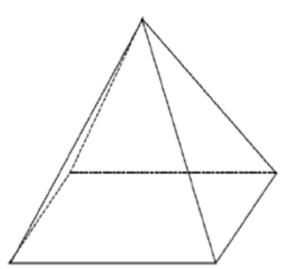


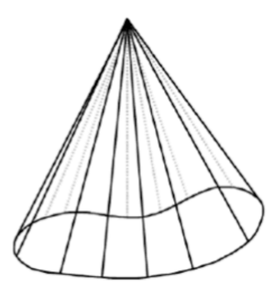
Here, maybe this from the module will help:

- **Cone** (Let B be a region in a plane E, and V be a point not in E. The *cone with base* B *and vertex* V is the union of all segments VP for all points P in B. If the base is a polygonal region, then the cone is usually called a *pyramid*.)









Go to the link below to see the geometric proof of the volume formula for cones and pyramids. It is a great visual video (in Spanish??) that shows how 3 pyramids make a prism and also shows Cavalieri's Principle. This can be done with paper pyramids as well.
https://www.youtube.com/watch?v=t3rOmcuioMk

Module lesson #7 page 104question
The cross sections of pyramids have a nice connection to dilation.

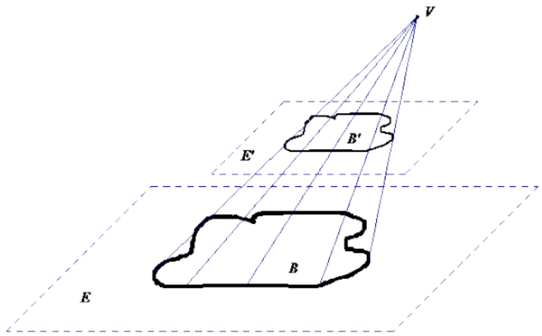
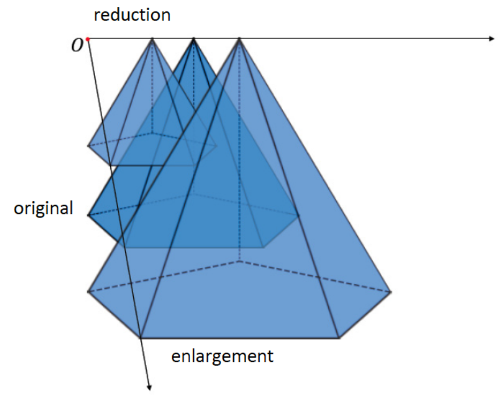


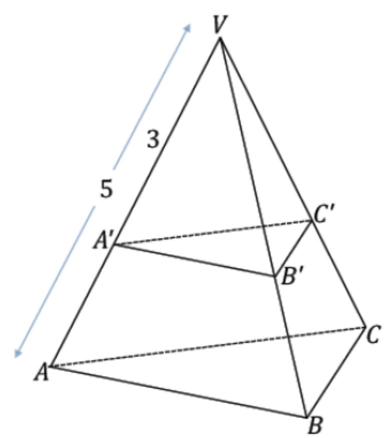
Figure 6



Module lesson #7 page 105 question
I'm hoping that a problem like the one below will not be asked since r:r^2:r^3 is not in the standards.

Example 1

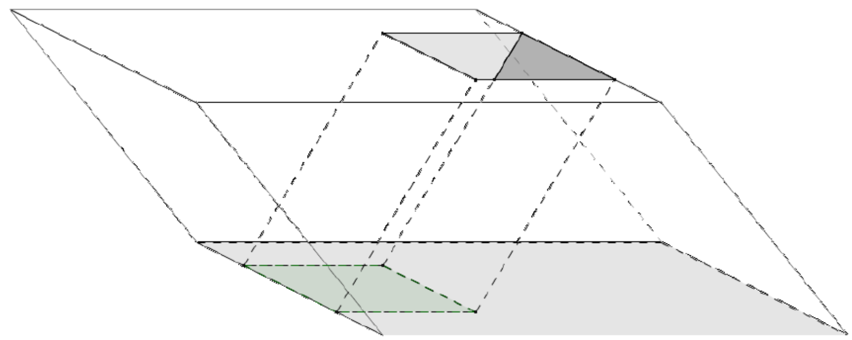
In the following triangular pyramid, a plane passes through the pyramid so that it is parallel to the base and results in the cross-section $\triangle A'B'C'$. If the area of $\triangle ABC$ is $25\ \mathrm{mm}^2$, what is the area of $\triangle A'B'C'$?



But – given the dimensions of the base, students should be able to use similarity to find the dimensions of the cross section and the volume of the top if asked. (See the next page)

Module lesson #10 page 157 question

3. An oblique prism has a rectangular base that is $16 \text{ in.} \times 9 \text{ in.}$ A hole in the prism is also the shape of an oblique prism with a rectangular base that is 3 in. wide and 6 in. long, and the prism's height is 9 in. (as shown in the diagram). Find the volume of the remaining solid.



Module lesson #11 page 169 question

Suppose you fill a conical paper cup with a height of $6"$ with water. If all the water is then poured into a cylindrical cup with the same radius and same height as the conical paper cup, to what height will the water reach in the cylindrical cup?

Module lesson #13 page 197 question

2. A cone with a radius of 5 cm and height of 8 cm is to be printed from a 3D printer. The medium that the printer will use to print (i.e., the "ink" of this 3D printer) is a type of plastic that comes in coils of tubing which has a radius of $1\frac{1}{3} \text{cm}$. What length of tubing is needed to complete the printing of this cone?

Module lesson #13 page 202 question

7. Filament for 3D printing is sold in spools that contain something shaped like a wire of diameter 3 mm. John wants to make 3D printings of a cone with radius 2 cm and height 3 cm. The length of the filament is 25 meters. About how many cones can John make?

8. John has been printing solid cones but would like to be able to produce more cones per each length of filament than you calculated in Problem 7. Without changing the outside dimensions of his cones, what is one way that he could make a length of filament last longer? Sketch a diagram of your idea and determine how much filament John would save per piece. Then determine how many cones John could produce from a single length of filament based on your design.

Expected level for spheres

Module lesson #12 page 187 question

8. The base of a circular cone has a diameter of 10 cm and an altitude of 10 cm. The cone is filled with water. A sphere is lowered into the cone until it just fits. Exactly one-half of the sphere remains out of the water. Once the sphere is removed, how much water remains in the cone?

End of module assessment

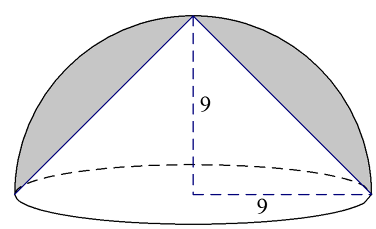
1. A machine part is manufactured from a block of iron with circular cylindrical slots. The block of iron has a width of *14* in., a height of *16* in., and a length of *20* in. The number of cylinders drilled out of the block is determined by the weight of the leftover block, which must be less than *1,000* lb.

    a. If iron has a weight of roughly 491 lb/ft3, how many cylinders with the same height as the block and with radius 2 in. must be drilled out of the block in order for the remaining solid to weigh less than 1,000 lb.?

    b. If iron ore costs *$115* per ton (*1* ton=*2200* lb.) and the price of each part is based solely on its weight of iron, how many parts can be purchased with *$1,500*? Explain your answer.

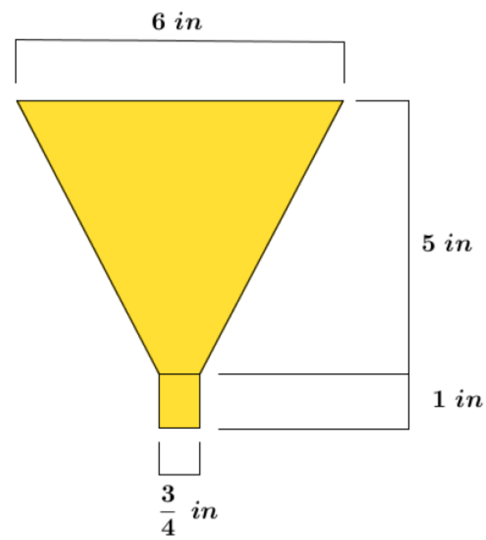
End of module assessment

2. In a solid hemisphere, a cone is removed as shown. Calculate the volume of the resulting solid. In addition to your solution, provide an explanation of the strategy you used in your solution.



Module lesson #11 page 171 question

12. A bulk tank contains a heavy grade of oil that is to be emptied from a valve into smaller 5.2-quart containers via a funnel. To improve the efficiency of this transfer process, Jason wants to know the greatest rate of oil flow that he can use so that the container and funnel do not overflow. The funnel consists of a cone that empties into a circular cylinder with the dimensions as shown in the diagram. Answer each question below to help Jason determine a solution to his problem.



a. Find the volume of the funnel.

b. If $1\ \text{in}^3$ is equivalent in volume to $\frac{4}{231}$ qt., what is the volume of the funnel in quarts?

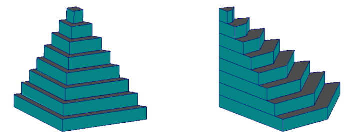
c. If this particular grade of oil flows out of the funnel at a rate of 1.4 quarts per minute, how much time in minutes is needed to fill the 5.2-quart container?

d. Will the tank valve be shut off exactly when the container is full? Explain.

e. How long after opening the tank valve should Jason shut the valve off?

f. What is the maximum constant rate of flow from the tank valve that will fill the container without overflowing either the container or the funnel?

(4) Cavalieri's Principle



Cavalieri's Principle (Given two solids that are included between two parallel planes, if every plane parallel to the two planes intersects both solids in cross-sections of equal area, then the volumes of the two solids are equal.)

Go to the websites below which demonstrate Cavalieri's Principle in 2D and 3D. Each video takes about 5 minutes to go through.

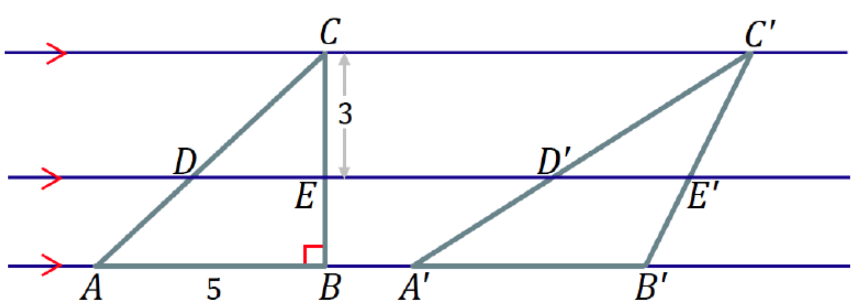
https://schoolyourself.org/learn/geometry/cavalieri-2d

https://schoolyourself.org/learn/geometry/cavalieri-3d

Write down any thoughts or questions:

Module lesson #10 page 150 question

The following triangles have equal areas: $\text{Area}(\triangle\, ABC) = \text{Area}(\triangle\, A'B'C') = 15 \text{ units}^2$. The distance between $\overleftrightarrow{DE}$ and $\overleftrightarrow{CC'}$ is 3. Find the lengths $\overline{DE}$ and $\overline{D'E'}$.

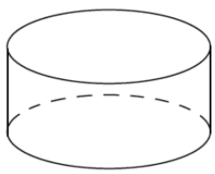
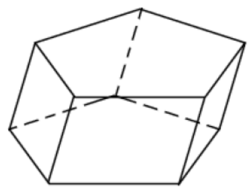
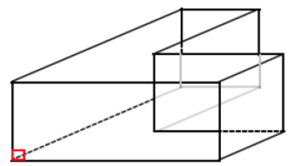
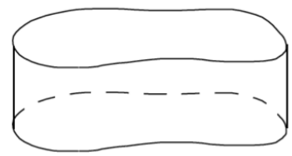


Joey says that if two figures have the same height and the same area, then their cross-sectional lengths at each height will be the same. Give an example to show that Joey's theory is incorrect.

Module lesson #13 page 193 question

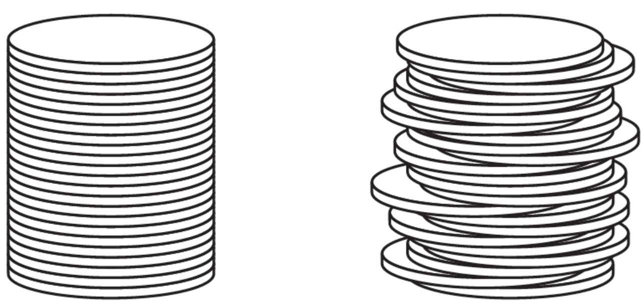
Recognizing cross sections is an important skill for Cavalieri's Principle. Try this problem from the module:

Sketch the cross-section for the following figures:

a. 	b. 	c. 	d. 

Released sample question

5 Two stacks of 23 quarters each are shown below. One stack forms a cylinder but the other stack does not form a cylinder.



Use Cavelieri's principle to explain why the volumes of these two stacks of quarters are equal.

Module lesson #13 page 193 question

Build a cone activity from lesson 13 about 3D printers

Lead students through a discussion that elicits how to build a right circular cone (such as the one in the Opening Exercise) with common materials, such as foam board, styrofoam, cardboard, and card stock and what must be true about the material to get a good approximation of the cone.

Begin by showing students a sheet of Styrofoam; for example, a piece with dimensions $10 \text{ cm} \times 10 \text{ cm} \times 1 \text{ cm}$. The idea is to begin with a material that is not as thin as say, card stock, but rather a material that has some thickness to it.

- Suppose I have several pieces of this material. How can I use the idea of slices to build the cone from the Opening Exercise? What steps would I need to take?

Allow students a moment to discuss with a partner before sharing out responses.

  - *You can cut several slices of the cone.*
  - *The disk cross-sections would have to be cut and stacked and aligned over the center of each disk. Each successive disk after the base disk must have a slightly smaller radius.*

- Say the cone we are trying to approximate has a radius of 3 inches and a height of 3 inches. Assume we have as many pieces of quarter-inch thick styrofoam as we want. How many cross-sections would there be, and how would you size each cross-section?

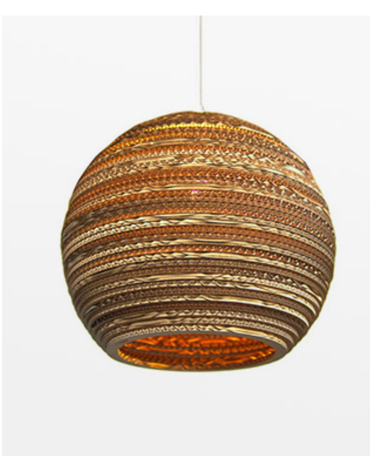
Allow students time to discuss and make the necessary calculations. Have them justify their responses by showing the calculations or the exact measurements.

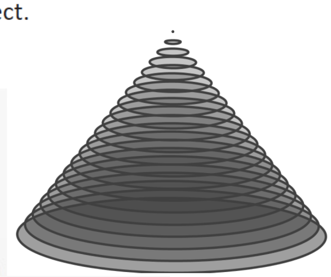
Now hold up a sheet of the styrofoam and a thinner material, such as card stock, side by side.

- If the same cone is built with both of these materials, which will result as a better approximation of the cone? Why?
  - *The use of card stock will result in a better approximation of the cone, since the thickness of the card stock would require more cross-sections, and sizing each successive cross-section would be more finely sized than the quarter-inch jumps in radius of the styrofoam.*
- To create a good approximation of a three-dimensional object, we must have ideal materials, ones that are very thin, and many cross-sections, in order to come as close as possible to the volume of the object.

Scaffolding:

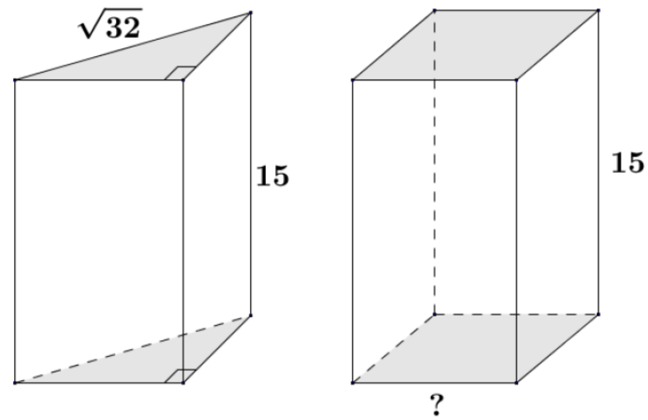
- Consider having a model of this example built to share with students after completing the exercise.





Module lesson #10 page 155 question

2 A triangular prism has an isosceles right triangular base with a hypotenuse of $\sqrt{32}$ and a prism height of 15. A square prism has a height of 15 and its volume is equal to that of the triangular prism. What are the dimensions of the square base?

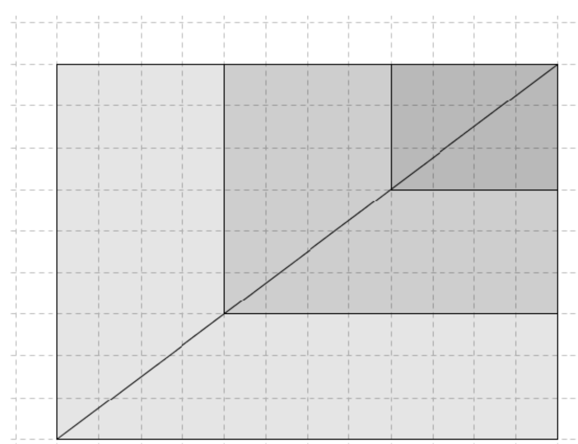


Module lesson #10 page 158 question

6. Use Cavalieri's principle to explain why a circular cylinder with a base of radius 5 and a height of 10 has the same volume as a square prism whose base is a square with edge length $5\sqrt{\pi}$ and whose height is also 10.

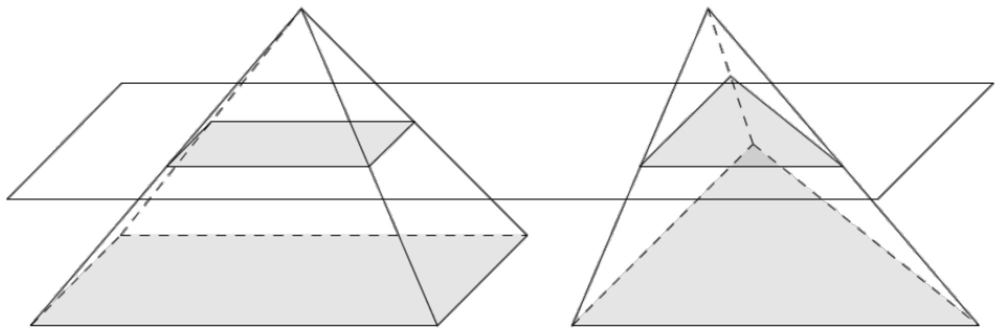
Module lesson #7 page 114 question

5 Liza drew the top view of a rectangular pyramid with two cross-sections as shown in the diagram and said that her diagram represents one, and only one, rectangular pyramid. Do you agree or disagree with Liza? Explain.



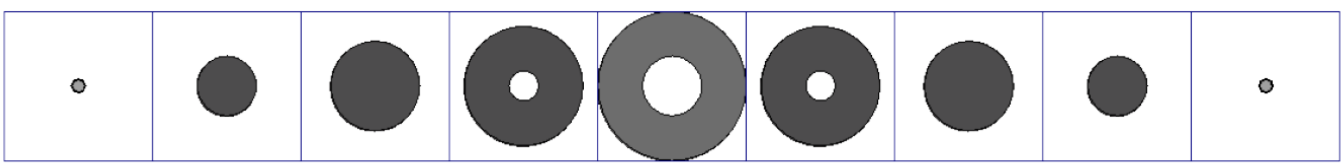
Module lesson #7 page 115 question

8 A rectangular cone and a triangular cone have bases with the same area. Explain why the cross-sections for the cones halfway between the base and the vertex have the same area.



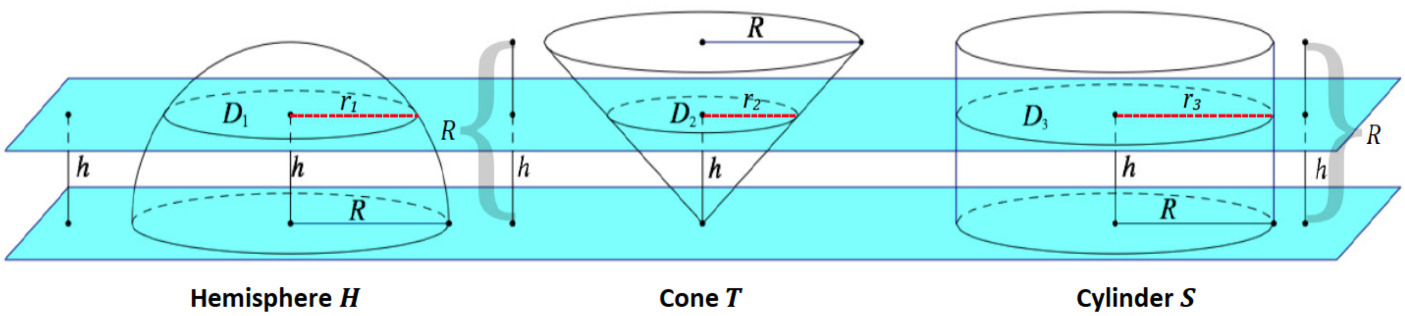
Module lesson #13 page 200 question

1. Horizontal slices of a solid are shown at various levels arranged from highest to lowest. What could the solid be?



Module lesson #12 page 175 question

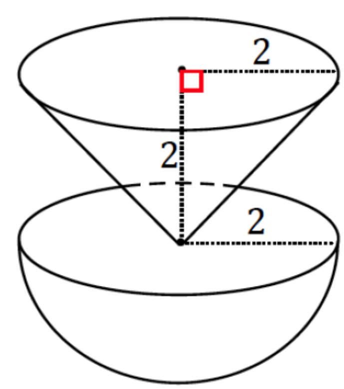
An extension standard worth looking at is using Cavalieri's to find the volume formula of a sphere. Below is a diagram from lesson 12. Also, the following link has a nice video explanation. https://www.youtube.com/watch?v=4cg8rPLa_sE



Module lesson #12 page 191 question

Challenge level for spheres

12. Challenge: An inverted, conical tank has a circular base of radius 2 m and a height of 2 m and is full of water. Some of the water drains into a hemispherical tank, which also has a radius of 2 m. Afterward, the depth of the water in the conical tank is 80 cm. Find the depth of the water in the hemispherical tank.



(5) Other Applications (including rotation of 2D shape to make 3D shape)

Released sample question

12 Trees that are cut down and stripped of their branches for timber are approximately cylindrical. A timber company specializes in a certain type of tree that has a typical diameter of 50 cm and a typical height of about 10 meters. The density of the wood is 380 kilograms per cubic meter, and the wood can be sold by mass at a rate of $4.75 per kilogram. Determine and state the minimum number of whole trees that must be sold to raise at least $50,000.

Released sample question

6 A contractor needs to purchase 500 bricks. The dimensions of each brick are 5.1 cm by 10.2 cm by 20.3 cm, and the density of each brick is $1920\ \text{kg/m}^3$.

The maximum capacity of the contractor's trailer is 900 kg. Can the trailer hold the weight of 500 bricks? Justify your answer.

End of module assessment

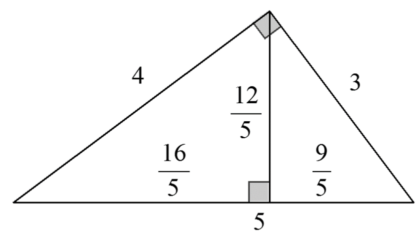
3.

a. A 3×5 rectangle is revolved about one of its sides of length *5* to create a solid of revolution. Find the volume of the solid.

b. A *3-4-5* right triangle is revolved about a leg of length *4* to create a solid of revolution. Describe the solid.

c. A *3-4-5* right triangle is revolved about its legs to create two solids. Find each of the volumes of the two solids created.

d. Show that the volume of the solid created by revolving a *3-4-5* triangle about its hypotenuse is 485π.



End of module assessment

4.

a. State the volume formula for a cylinder. Explain why the volume formula works.

b. The volume formula for a pyramid is $13Bh$, where B is the area of the base and h is the height of the solid. Explain where the *13* comes from in the formula.

c. Give an explanation of how to use the volume formula of a pyramid to show that the volume formula of a circular cone is $1/3\ \pi r2h$, where r is the radius of the cone and h is the height of the cone.

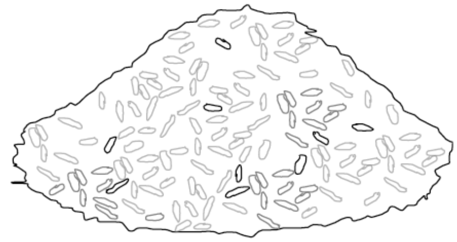
End of module assessment

5. A circular cylinder has a radius between *5.50* and *6.00* centimeters and a volume of *225* cubic centimeters. Write an inequality that represents the range of possible heights the cylinder can have to meet this criterion to the nearest hundredth of a centimeter.

End of module assessment

Gold has a density of $19.32\ \text{g/cm}^3$. If a square pyramid has a base edge length of $5\ \text{cm}$, height of $6\ \text{cm}$, and a mass of $942\ \text{g}$, is the pyramid in fact solid gold? If it is not, what reasons could explain why it is not? Recall that density can be calculated with the formula density $= \frac{\text{mass}}{\text{volume}}$.

End of module assessment



6. Rice falling from an open bag piles up into a figure conical in shape with an approximate radius of 5 cm.

   a. If the angle formed by the slant of the pile with the base is roughly 30°, write an expression that represents the volume of rice in the pile.

   b. If there are approximately *20* grains of rice in a cubic centimeter, approximately how many grains of rice are in a *4.5*-kilogram bag of rice?

End of module assessment

7. Describe the shape of the cross-section of each of the following objects.

Right circular cone:

a. Cut by a plane through the vertex and perpendicular to the base

Square pyramid:

b. Cut by a plane through the vertex and perpendicular to the base

c. Cut by a vertical plane that is parallel to an edge of the base but not passing through the vertex

Sphere with radius r:

d. Describe the radius of the circular cross-section created by a plane through the center of the sphere.

e. Describe the radius of the circular cross-section cut by a plane that does not pass through the center of the sphere.

Triangular Prism:

f. Cut by a plane parallel to a base

g. Cut by a plane parallel to a face

Closing:

What TO spend time on
1) Application problems
2) Cavalieri's principle
3) Dissection
4) area as a limit

What NOT to spend time on (IMHO)
1) $r:r^2:r^3$ length, area, volume similarity
2) reference to sets (unless your students already know it
3) surface area (grade 6 and 7 standards)

Table of Contents[1]

Extending to Three Dimensions

[1] Each lesson is ONE day, and ONE day is considered a 45-minute period.

3DPD for CCSS Geometry with Lara Lomac (lara.lomac@rcsdk12.org) http://rcsdk12.org//Domain/6881 January 21, 2015

Additional resources

•Nrich growing rectangles (http://nrich.maths.org/6923)
•Mathshell Best Cans (http://map.mathshell.org/materials/tasks.php?taskid=284&subpage=expert)
•Mathshell Propane tanks (http://map.mathshell.org/materials/tasks.php?taskid=288&subpage=expert)
•Mathshell Fruit Boxes (optimize volume given area)
(http://map.mathshell.org/materials/tasks.php?taskid=288&subpage=expert)
•Mathshell Fearless Frames (optimize volume given length)
(http://map.mathshell.org/materials/tasks.php?taskid=277&subpage=expert)
•Mathshell Security Camera (area)
(http://map.mathshell.org/materials/tasks.php?taskid=273&subpage=expert)
•Mathshell Glasses (volume complex shape and find height)
(http://map.mathshell.org/materials/tasks.php?taskid=259&subpage=apprentice)

Learnzillion (start at 1:08) https://learnzillion.com/lessons/3488-predict-3d-results-of-rotating-simple-figures

making prisms in a computer program https://www.youtube.com/watch?v=doVFikdRid8

Iron Cad spin to make a part https://www.youtube.com/watch?v=RFFCUEJWiSA

Cavalieri's Principal from school yourself 2d https://schoolyourself.org/learn/geometry/cavalieri-2d
Cavalieri's Principal from school yourself 3d https://schoolyourself.org/learn/geometry/cavalieri-3d

sketchup (pull-push 2d into 3d) https://www.youtube.com/watch?v=dy81IdYQdgY

Connection to calculus https://www.youtube.com/watch?v=bw23lWXpAlc

Khan rotate 2d shape in 3d (not great)
https://www.khanacademy.org/math/geometry/basic-geometry/cross-sections/v/rotating-2d-shapes-in-3d

Sphere https://www.youtube.com/watch?v=4cg8rPLa_sE

3 pyramids make a prism and cavalieri's https://www.youtube.com/watch?v=t3rOmcuioMk

Cavalieri's principal from Wolfram http://demonstrations.wolfram.com/CavalierisPrinciple/
Solids by revolution from Wolfram http://demonstrations.wolfram.com/SolidsOfRevolution/

Rotate a triangle http://www.geogebratube.org/student/m14914
Reimann sums from Geogebra http://www.geogebratube.org/student/m4897
Transforming volume prism from Geogebra http://www.geogebratube.org/student/m78861
Truncated cone revolution from Geogebra http://www.geogebratube.org/student/m40920
Cone and cylinder revolution from Geogebra http://www.geogebratube.org/student/m124734

Cavalieri's Principle formal definition
http://www.cut-the-knot.org/Curriculum/Calculus/Cavalieri.shtml

Cylinder Definitions from various sources

(1) NYS Mathematics Glossary* – Geometry

right circular cylinder A cylinder whose bases are circles and whose altitude passes through the center of both bases.

(2) regents prep **http://www.regentsprep.org/regents/math/geometry/gg2/CylinderPage.htm**

Cylinders are three-dimensional closed surfaces.

In general use, the term *cylinder* refers to a right circular cylinder with its ends closed to form two circular surfaces, that lie in parallel planes.

Cylinders are **not** called polyhedra since their faces are not polygons. In many ways, however, a cylinder is similar to a prism. A cylinder has **parallel congruent bases**, as does a prism, but the cylinder's bases are circles rather than polygons.

(3) http://quizlet.com/20117041/holt-geometry-chapter-ten-vocabulary-flash-cards/

a three-dimensional figure with two parallel congruent circular bases and a curved lateral that connects the bases

(4) *http://www.mathopenref.com/cylinder.html*

A cylinder is a closed solid that has two parallel (usually circular) bases connected by a curved surface.

(5) *http://www.merriam-webster.com/dictionary/cylinder*

a : the surface traced by a straight line moving parallel to a fixed straight line and intersecting a fixed planar closed curve

b : a solid or surface bounded by a cylinder and two parallel planes cutting all its elements; *especially* **:** right circular cylinder — see volume table

(6) *http://dictionary.reference.com/browse/cylinder*

Geometry. a surface or solid bounded by two parallel planes and generated by a straight line moving parallel to the given planes and tracing a curve bounded by the planes and lying in a plane perpendicular or oblique to the given planes.

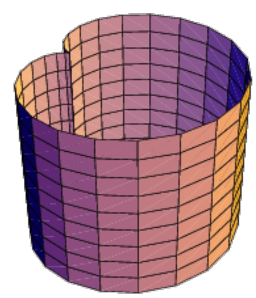
(7) https://www.mathsisfun.com/definitions/cylinder.html

A solid object with:

• two identical flat ends that are circular or elliptical

• and one curved side.

It has the same cross-section from one end to the other.



(8) http://mathworld.wolfram.com/GeneralizedCylinder.html

A ruled surface is called a generalized cylinder if it can be parameterized by , where is a fixed point. A generalized cylinder is a regular surface wherever . The above surface is a generalized cylinder over a cardioid. A generalized cylinder is a flat surface, and is sometimes called a "cylindrical surface" (Kern and Bland 1948, p. 32) or "cylinder surface" (Harris and Stocker 1998, p. 102).

A generalized cylinder need not be closed (Kern and Bland 1948, p. 32).

Kern and Bland (1948, p. 32) define a cylinder as a solid bounded by a generalized cylinder and two parallel planes. However, when used without qualification, the term "cylinder" generally refers to the particular case of a right circular cylinder.

Some history of pi.
Egyptian 1650 BC Ahmes *pi* = 4(8/9)2 = 3.16049 men were stretching ropes in order to mark the property limits and areas for temples

Old Testament In Hebrew, each letter equals a certain number, and a word's "value" is equal to the sum of its letters. Interestingly enough, in 1 Kings 7:23, the word "line" is written Kuf Vov Heh, but the Heh does not need to be there, and is not pronounced. With the extra letter , the word has a value of 111, but without it, the value is 106. (Kuf=100, Vov=6, Heh=5). The ratio of *pi* to 3 is very close to the ratio of 111 to 106. In other words, pi/3 = 111/106 approximately; solving for pi, we find *pi* = 3.1415094... (Tsaban, 78). This figure is far more accurate than any other value that had been calculated up to that point, and would hold the record for the greatest number of correct digits for several hundred years afterwards. Unfortunately, this little mathematical gem is practically a secret, as compared to the better known *pi* = 3 approximation

Greeks took up the problem, they took two revolutionary steps to find pi. Antiphon and Bryson of Heraclea came up with the innovative idea of inscribing a polygon inside a circle, finding its area, and doubling the sides over and over . "Sooner or later (they figured), ...[there would be] so many sides that the polygon ...[would] be a circle"

Archimedes took up the challenge. However, he used a slightly different method than they used. Archimedes focused on the polygons' perimeters as opposed to their areas, so that he approximated the circle's circumference instead of the area. He started with an inscribed and a circumscribed hexagon, then doubled the sides four times to finish with two 96-sided polygons.

China Tsu Ch'ung-chih and his son Tsu Keng-chih came up with astonishing results, when they calculated 3.1415926 < *pi* < 3.1415927. The father and son duo used inscribed polygons with as many as 24,576 sides. (Blatner, 25) Soon after, the Hindu mathematician Aryabhata gave the 'accurate' value 62,832/20,000 = 3.1416 (as opposed to Archimedes' 'inaccurate' 22/7 which was frequently used)

In the twentieth century, computers took over the reigns of calculation, and this allowed mathematicians to exceed their previous records to get to previously incomprehensible results. In 1945, D. F. Ferguson discovered the error in William Shanks' calculation from the 528th digit onward. Two years later, Ferguson presented his results after an entire year of calculations, which resulted in 808 digits of pi. (Berggren, 406)